Kardar-Parisi-Zhang equation and the δ expansion

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We show how the δ expansion of Bender *et al.* can be used as a relevant tool for the study of the Kardar-Parisi-Zhang equation. We obtain the exponents for the substrate dimension $D=2$ and the critical value of D for which the weak coupling exponents appear.
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The question of kinetic roughening of interfaces has attracted immense attention in the last decade. A much studied paradigm is the Kardar-Parisi-Zhang (KPZ) model [1]. For substrate dimensions (D) below two, it shows a roughening of the interface. Less well understood is the situation for $D\geqslant 2$, where the model is supposed to show a transition from rough to smooth behavior as the strength of the coupling constant of the nonlinear term is varied. Since the perturbative renormalization group treatment works only for $D<2$, various new techniques [2,3] have been introduced to deal with the situation for $D\geqslant 2$. In this paper we propose an amusing, though not very accurate, trick for looking at this problem. We exploit the δ expansion technique of Bender *et al.* [4]. To our knowledge, the δ expansion has never been employed for studying nonlinear stochastic differential equations. The reason, we believe, lies in the technical complication involved in doing a genuine second order δ-expansion calculation. We circumvent this problem by making use of an approximation. Another important issue in the δ-expansion calculation is the choice of the unperturbed theory. In the study of critical exponents by this technique, it was noted by Gandhi and McKane [5] that the starting point is vital. This has been a crucial point for us as well.

The height $h(\vec{r},t)$ of the interface above some reference level has the stochastic dynamics

$$\frac{\partial h}{\partial t}=\Gamma\nabla^2 h+\lambda(\vec{\nabla}h)^2+N, \tag{1a}$$

$$\langle N(\vec{r},t)N(\vec{r}',t')\rangle=2D_0\delta(\vec{r}-\vec{r}')\,\delta(t-t') \tag{1b}$$

in the KPZ picture. The vector $\vec{r}$ is a D-dimensional vector in the substrate. The quantities that are usually studied are (i) the response or Green's function $G(k,w)$,

$$G(k,w)=\frac{1}{\delta^D(\vec{k}+\vec{k}')\,\delta(w+w')}\left\langle\frac{\partial h(k,w)}{\partial N(k',w')}\right\rangle \tag{2}$$

and (ii) the correlation function $C(k,w)$

$$C(k,w)=\frac{\langle h(k,w)h(k',w')\rangle}{\delta^D(\vec{k}+\vec{k}')\,\delta(w+w')}\,. \tag{3}$$

A scaling solution is assumed for $G(k,w)$ and $C(k,w)$, namely,

$$G(k,w)=k^{-z}g(w/k^z) \tag{4a}$$

and

$$C(k,w)=\frac{1}{k^{D+2\alpha+z}}\,f(w/k^z). \tag{4b}$$

The exponent z has to do with critical slowing down and the exponent α has to do with roughness. The interface is rough if $\alpha>0$. Since the perturbative renormalization group works very well for $D<2$, the questions one would like to address in any new scheme are as follows: (i) What is the value of z for $D=2$? (ii) At what value of D_c (upper critical dimension) of D is $z=2$, the mean field value?

The idea behind our δ expansion is to consider the dynamics

$$\frac{\partial h}{\partial t}=(-\nabla^2)^{\delta}[\Gamma\nabla^2 h+\lambda(\vec{\nabla}h)^{2+2\delta}]+N, \tag{5a}$$

$$\langle N(\vec{r}',t')N(\vec{r},t)\rangle=2D_0\delta(\vec{r}-\vec{r}')\,\delta(t-t'). \tag{5b}$$

We note that for $\delta=0$ one has the KPZ problem, while for $\delta=1$, we have a problem that can be exactly solved [6] for $1<D<4$. Our method will be to start from $\delta=0$ with unknown exponents α and z and produce the lowest order nontrivial correction from the δ expansion. We match the resulting α and z for $\delta=1$ to the exact answer. Thus the KPZ values for the exponents are obtained.

It should be pointed out that our approach is different from the traditional one. In the usual method of doing perturbation theory, one starts from an exactly solvable problem and builds up corrections to the exact answer. In this case, we find it more convenient to start from the problem that cannot be exactly solved (KPZ), find the first correction to it in perturbation theory, and match the resulting answer to an exactly solvable case.

We now explain how the δ expansion is supposed to work and what technical problem is going to arise. Turning to Eq. (5a), we note that $[(\vec{\nabla}h)^2]^{1+\delta}$ can be expanded as

$$[(\vec{\nabla}h)^2]^{1+\delta}=(\vec{\nabla}h)^2\left[1+\delta\,\ln(\vec{\nabla}h)^2+\frac{\delta^2}{2}\,[\ln(\vec{\nabla}h)^2]^2+\cdots\right]. \tag{6}$$

If we introduce the expansion of Eq. (6) into Eq. (5) and attempt a perturbation theory in δ, the calculation will be

very awkward. Instead, one studies the theory with the polynomial interaction [4] written as

$$\frac{\partial h}{\partial t}=(-\nabla^2)^{\delta}\left[\Gamma(\nabla^2 h)+\lambda(\vec{\nabla}h)^2+\lambda\sum_{k=1}^{K}(\vec{\nabla}h)^{2+\beta_k}P_k\right]+N, \tag{7}$$

where the coefficients P_k are polynomials in $\beta_1,\beta_2,\ldots,\beta_k$ and δ. To begin with, one considers the β_i to be integers so that the usual diagrammatics can be performed. The Green's function $\tilde{G}$ of the system with equation of motion given by Eq. (7) can now be calculated. What one requires, however, is the Green's function G of the system described by Eqs. (5) and (6). To get G from $\tilde{G}$, we exploit the identity $(d/dx)a^x=a^x \ln a$, consider the set $\{P_i\}$ as continuous, choose the correct differential operator D,

$$D=\sum_{j,k} A_{jk}\left[\frac{\partial}{\partial\beta_k}\right]^j, \tag{8}$$

and finally use

$$G=D\tilde{G}|_{\beta_1=\beta_2=\cdots=\beta_k=0}. \tag{9}$$

To work to $O(\delta^2)$, we need

$$P_1=\delta+\delta^2,\quad P_2=-\delta+\delta^2,$$
$$D_2=\frac{1}{2}\left[\frac{\partial}{\partial\beta_1}-\frac{\partial}{\partial\beta_2}\right]+\frac{1}{4}\left[\frac{\partial^2}{\partial\beta_1^2}+\frac{\partial^2}{\partial\beta_2^2}\right]. \tag{10}$$

The strategy is now plain. If the response function $G(k,w)$ is to be calculated, then we first evaluate $\tilde{G}(k,w)$ using Dyson's equation, $\tilde{G}^{-1}(k,w)=G_0^{-1}(k,w)+\tilde{\Sigma}(k,w)=-iw+\Gamma_0(k)+\tilde{\Sigma}(k,w)$, where $G_0^{-1}(k,w)=-iw+\Gamma_0(k)$ is the response for the KPZ system and $\tilde{\Sigma}(k,w)$ is the self energy for the polynomial interaction of Eq. (7). The task is to calculate $\tilde{\Sigma}(k,w)$ as a function of β_k (one calculates for integer valued β_k and generalizes) and use the derivative operator of Eq. (10) to arrive at $\Sigma(k,w)$ and thus $G(k,w)$. The difficulty is also clear at this point. For different values of β_k an actual analytic expression has to be obtained for $\tilde{\Sigma}(k,w)$, so that the derivative may be evaluated when necessary. The analytic evaluation of $\tilde{\Sigma}(k,w)$ for arbitrary integer values k is near impossible. We now explain the approximation that lets us proceed further.

Returning to Eq. (7), we note that for $\beta_k=1$, we have the usual KPZ-like term with the single loop self-energy given by [of necessity the lowest order calculation is $O(\delta^2)$ and hence P_2 is what we are interested in]

$$\sum(k,w=0)=\lambda^2 P_2^2\int_{\vec{p}+\vec{q}=\vec{k}}\frac{d^D p}{(2\pi)^D}\frac{dw}{(2\pi)}\times(\vec{p}\cdot\vec{k})(\vec{p}\cdot\vec{q})C(\vec{p},w)G(\vec{q},-w), \tag{11}$$

where $C(p,w)$ and $G(q,w)$ stand for the correlation function and response function of the system with $P_k=0$ and in the contribution to $\Sigma(k)$, δ has been set equal to zero in the integrand since P_2^2 is already $O(\delta^2)$. Now, for $\beta_k=2n$, where n is an integer,

$$\tilde{\Sigma}_n(k,w=0)=\lambda^2 P_2^2\int_{\vec{p}_1+\vec{p}_2+\cdots\vec{p}_{2n+1}+\vec{q}}\prod_{i=1}^{2n}\frac{d^D p_i}{(2\pi)^D}\frac{dw_i}{2\pi}(\vec{q}\cdot\vec{p}_1)(\vec{k}\cdot\vec{p}_1)(\vec{p}_2\cdot\vec{p}_3)^2\cdots(\vec{p}_{2n}\cdot\vec{p}_{2n+1})^2$$
$$\times C(\vec{p}_1,w_1)C(\vec{p}_2,w_2)\cdots C(\vec{p}_{2n+1},w_{2n+1})G\left(\vec{q},-\sum_i w_i\right)(2n+2)(2n+2)! \tag{12}$$

One needs to evaluate $\tilde{\Sigma}(k)$ of Eq. (12) as an analytic function of n, whence it becomes a function of β_k and allows for the evaluation of the derivative. The task of obtaining an analytic expression for Eq. (12) for arbitrary n is clearly hopeless.

It would be possible to obtain an analytic approximant for $\tilde{\Sigma}_n(k)$ if in some limit the different integrals factored. To get the required insight, we return to Eq. (11) and note that $\Sigma(k)$ can be expanded in a power series about its various poles [7]. The particular pole, which is of the greatest help to us, is the one that obtains from the region of the integrand with $p\simeq 0$. With the scalings of Eqs. (4a) and (4b), the pole contribution is obtained as

$$\Sigma_{\text{pole}}=\lambda^2 k^{2-z}P_2^2\frac{C_D}{(2\pi)^D}\frac{1}{D}\int_{p\leqslant k}\frac{p^{D+1}dp}{p^{D+2\alpha}}$$
$$=\lambda^2 k^{2-z}P_2^2\frac{C_D}{(2\pi)^D}\frac{1}{D}A\frac{k^{2(1-\alpha)}}{2(1-\alpha)}, \tag{13}$$

where A is a constant involving the amplitudes of the correlation function and relaxation rate. To make use of the same pole approximation in Eq. (12), we note that the required part of the phase space of the integrand comes from the region $p_1,p_2,\ldots,p_{2n+1}\simeq 0$, $q\simeq k$. Now, the different moments are decoupled and the pole approximation leads to the factor $(1-\alpha)^{-2n}$. Taking the combinational factor into account,

$$\tilde{\Sigma}_n = \Sigma_{\rm pole} \frac{(2n+2)!(2n+2)}{4} M^{2n} = \Sigma_{\rm pole} \frac{M^\beta \Gamma(\beta+3)}{4}, \tag{14}$$

where

$$M = A \frac{C_D}{(2\pi)^D} \frac{1}{D} \frac{k^{2(1-\alpha)}}{2(1-\alpha)}.$$

To obtain the self-energy Σ of the original problem, one needs to operate on $\tilde{\Sigma}_n$ with D_2 of Eq. (10) and this leads to

$$\Sigma(k) = \Sigma_{\rm pole} \left\{ \ln^2 M + [2 + \tfrac{4}{3}\sqrt{3}\Gamma'(3)] \times \ln M + \frac{4\sqrt{3}}{3} [\Gamma'(3) + \Gamma''(3)] \right\}. \tag{15}$$

The self-energy $\Sigma(k)$ is simply the change $\Delta\Gamma$ in the relaxation rate due to finite δ and in the approximation made,

$$\Sigma_{\rm pole} = \delta^2 \Gamma_0(k),$$

where $\Gamma_0(k)$ is the relaxation rate of the pure KPZ (i.e., $\delta=0$). Thus

$$\frac{\Delta\Gamma(k)}{\Gamma_0(k)} = \delta^2 \left[\ln^2 M + \left(2 + \frac{4\sqrt{3}}{3} \Gamma'(3) \right) \times \ln M + \frac{4\sqrt{3}}{3} [\Gamma'(3) + \Gamma''(3)] \right]. \tag{16}$$

At this point, we need to discuss the accuracy of $\Delta\Gamma/\Gamma_0$ in Eq. (16), so far as the approximations made in evaluating the integrals for $\Delta\Gamma$ and Γ_0 are concerned. To do so, we consider the technique of Bouchaud and Cates [2] for determining the exponent α. From the self-energy and the correlation function, they obtain two integrals I and J as functions of α and D. The exponent α is obtained from the requirement that $I=J$. The integrals are

$$I = \int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p}\cdot(\vec{1}-\vec{p})][\vec{1}\cdot\vec{p}]}{p^{D+2\alpha}[|p|^{2-\alpha} + |\vec{1}-\vec{p}|^{2-\alpha}]}, \tag{17a}$$

$$J = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \frac{[\vec{p}\cdot(\vec{1}-\vec{p})]^2}{p^{D+2\alpha}|\vec{1}-\vec{p}|^{D+2\alpha}[|p|^{2-\alpha} + |\vec{1}-\vec{p}|^{2-\alpha}]}. \tag{17b}$$

The integrals are evaluated numerically and $I=J$ yields $\alpha = \frac{1}{2}$ for $D=1$, $\alpha=0.30$ for $D=2$, and $\alpha=0$ at $D_c=3.6$. If we evaluate I and J in the pole approximation, then we obtain $\alpha=(4-D)/6$, giving $\alpha=\frac{1}{2}$ at $D=1$, $\alpha=1/3$ at $D=2$, and $\alpha=0$ for $D_c=4$, in very good agreement with the numerical values. The individual integrals can differ by up to 20%, but the ratio always turns out to be quite accurate. We have investigated other situations (mainly in dynamic critical phenomena) and found that ratios turn out within 5% of the numerical answer in all the cases studied. We estimate the error in $\Delta\Gamma/\Gamma_0$ to be no more than 10% (the maximum error in the comparison shown above) due to the approximate evaluation of integrals.

Returning to Eq. (16), we note that $M \propto k^{2(1-\alpha_0)}$ (α_0 is the KPZ exponent) and hence $\Delta\Gamma$ has a constant $\ln k$ and $(\ln k)^2$ parts. The constant simply renormalizes the amplitude of the KPZ relaxation rate. The $\ln^2 k$ term will cancel with a similar term coming from the two-loop, i.e., $O(\lambda^4)$, contribution of the pure KPZ, forcing in the process a fixed point value of λ^2. This is equivalent to removing transients to find the asymptotic scaling behavior. The $\ln k$ term renormalizes the KPZ relaxation, producing a change Δz in the KPZ exponent. Thus, if the KPZ relaxation rate is $\Gamma_0(k) = \Gamma_0 k^{z_0}$ then to $O(\delta^2)$, the system given in Eq. (5a) has the exponent $z = z_0 + \Delta z$, where

$$\Delta z = \delta^2(1-\alpha_0)\left(1 + \frac{2\sqrt{3}}{3}\Gamma'(3)\right) = \delta^2(z_0-1)\left(1 + \frac{2\sqrt{3}}{3}\Gamma'(3)\right). \tag{18}$$

For $\delta=1$, the system given in Eq. (5a) has the exponent $z = \frac{3}{5}D + 2.4$ and thus

$$z_0 + (z_0-1)\left(1 + \frac{2\sqrt{3}}{3}\Gamma'(3)\right) = \tfrac{3}{5}D + 2.4, \tag{19}$$

the central result of this paper. This leads to (i) $z_0=1.67$ and $D=2$ and (ii) $D_c=4.07$, where $z_0=2$.

The values of z_0 and D_c are reasonably close to the numerical values ($z_0=1.63$ and $D_c=3.67$). However, that is more accidental than anything else. One would not advocate this scheme for its numerical accuracy; it is simply an amusing application of the δ expansion.

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[1] M. Kardar, G. Parisi, and Y. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[2] J. P. Bouchaud and M. E. Cates, Phys. Rev. E **47**, R1455 (1993).
[3] J. P. Doherty, M. A. Moore, J. M. Kim, and A. J. Bray, Phys. Rev. Lett. **72**, 2041 (1994).
[4] C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky, and L. M. Simmons, Phys. Rev. Lett. **58**, 2615 (1987); Phys. Rev. D **37**, 1472 (1988).
[5] S. K. Gandhi and A. J. McKane, Nucl. Phys. **B419**, 424 (1994).
[6] Z. W. Lai and S. Das Sarma, Phys. Rev. Lett. **66**, 325 (1993).
[7] J. K. Bhattacharjee and R. A. Ferrell, J. Math. Phys. (N.Y.) **21**, 534 (1980).